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NOTES ON A PROBLEM INVOLVING PERMUTATIONS  
AS SUBSEQUENCES

Malcolm Newey

Stanford University

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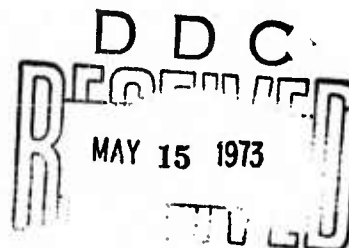
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MALCOLM NEWHEY

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ABSTRACT:

The problem (attributed to R. M. Karp by Knuth ( see #36 of [1] )) is to describe the sequences of minimum length which contain, as subsequences, all the permutations of an alphabet of  $n$  symbols. This paper catalogs some of the easy observations on the problem and proves that the minimum lengths for  $n=5$ ,  $n=6$  &  $n=7$  are 19, 28 and 39 respectively. Also presented is a construction which yields (for  $n \geq 2$ ) many appropriate sequences of length  $n^2 - 2n + 4$  so giving an upper bound on length of minimum strings which matches exactly all known values.

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## KEY WORDS

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# 1 NOTATION.

= =====

- a) Let  $S$  be a sequence of symbols.  $|S|$  will be used to denote the total number of symbols in  $S$  and so we observe, for example,  $|x y x z| = 4$ .
- b) We say  $x \subset y$  in the case where  $x$  is a subsequence of  $y$  and we say " $x$  is equivalent to  $y$ " if  $x$  can be obtained from  $y$  by a simple change of alphabet; we denote this equivalence by  $\sim$ .  
(e.g.  $x y \subset x y y x$ ,  $x y z x \sim 1 2 3 1$ )
- c)  $P(A)$  is used to denote the set of sequences which are permutations of an alphabet  $A$ . Cardinality of  $P(A)$  will be  $(|A|)!$ . Also,  $P'(A,n)$  is the set of permutations of all sub-alphabets of  $A$  of size  $n$  (where  $n \leq |A|$ ). Clearly,  $P(A) = P'(A, |A|)$ .
- d) If  $A$  is an alphabet then  $Q(A) = \{x \mid x \in A' \wedge \forall y. (y \in P(A) \supset y \subset x)\}$  where  $A'$  is the set of sequences over alphabet  $A$ . For example,  $abcacba \in Q(abc)$ . Also,  $Q'(A,n)$  is taken to be the set  $\{x \mid x \in A' \wedge \forall y. (y \in P(A,n) \supset y \subset x)\}$ . So, for example,  $zyxwxyz \in Q'(wxyz, 2)$ .
- e) Now, the LENGTHS of the shortest sequences in  $Q(A)$  and  $Q'(A,n)$  depend only on the SIZE of the alphabet  $A$ . Hence, take  $M(n)$  to be the length of the shortest sequence in  $Q(1 2 3 \dots n)$  and  $M'(n,m)$  to be the length of the shortest sequence in  $Q'(1 2 3 \dots n, m)$ .  
So, for example,  $M(1)=1$ ,  $M(2)=3$  and  $M'(n,1)=n$ .
- f)  $S(n)$  denotes the  $n$ -th symbol of sequence  $S$ .  
 $S(n:m)$  denotes that contiguous subsequence of sequence  $S$  which is the symbols from position number  $n$  in  $S$  to position number  $m$ .  
 $\#(S,x)$  denotes the number of occurrences of the symbol  $x$  in sequence  $S$ .
- g) "CPAF  $X$ " is just an abbreviation for "Consider the Permutations of the current Alphabet of the Form  $X$ ". The greek letters which appear in  $X$  denote arbitrary sequences of symbols.  
For example, if the alphabet under discussion were  $abcde$ , the command "CPAF  $bxc$ " would mean "Consider Permutations of  $abcde$  which start with  $b$  and end with  $c$ ".

## 2 SOME EASY OBSERVATIONS.

2.1  $M(1)=1.$

2.2  $M(2)=3.$

2.3  $M(3)=7.$

2.4  $M'(n,1)=n.$

2.5  $M'(n,2)=(2n-1)$  can be seen as follows:

$M'(n,2) \leq 2n-1$  since if  $A$  is an alphabet of length  $n$ , then the sequence  $AA(2:2n)$  is a member of  $Q'(A,2)$ .

$M'(n,2) \geq 2n-1$  since if  $A$  is an alphabet of size  $n$ ,  $S$  is a member of  $Q'(A,2)$  and  $|S| < 2n-1$  then at least two of the symbols of  $A$  ( $x$  and  $y$ , say) only appear once in  $S$ ; hence 1 of the sequences ' $xy$ ' and ' $yx$ ' are not subsequences of  $S$ .

2.6  $M'(n,m) \geq (m \cdot (2n-m+1))/2$  ( $n \geq m$ , of course)

This result is more easily remembered as

$$M'(n,m) \geq n + n-1 + n-2 + \dots + n-m+1.$$

Suppose  $A$  is an alphabet of size  $n$  and  $S$  is a sequence from  $Q'(A,m)$  of minimum length (i.e.  $|S|=M'(n,m)$ ). It is noted in (2.4) that  $M'(n,1)=n$  so take  $m \geq 2$ . Segment  $S$  as  $TxU$  where the sequences  $T,U$  and the symbol  $x$  are chosen so that  $x$  does not appear in  $T$  but all the other symbols of  $A$  do. Clearly,  $|T| \geq (n-1)$ . Now note that all permutations of subalphabets of  $A$  of size  $m$  which start with  $x$  are subsequences of  $xU$ . Hence all permutations of subalphabets of  $A \setminus x$  of size  $(m-1)$  are subsequences of  $U$  ( $A \setminus x$  is  $A$  without  $x$  and  $|A \setminus x| = (n-1)$ ).  $|U| \geq M'(n-1,m-1)$ , therefore, and so  $M'(n,m)$  (which is simply  $|S|$ ) is at least  $(n-1) + 1 + M'(n-1,m-1)$ . This recurrence relation is readily solved to give the result.

2.7  $M(n) \geq (n \cdot (n+1))/2.$

Simple corollary of 2.6 using  $M(n)=M'(n,n)$ .



$$2.8 \quad M'(n,m) \leq (m \cdot (n-1) + 1)$$

Given an alphabet, A, of size n, the following construction gives an element of  $Q'(A,m)$  of length  $m \cdot (n-1) + 1$  :-  
Generate m permutations of the alphabet  $A_1, A_2, A_3, \dots, A_m$  such that  $A_1(n) = A_2(1), A_2(n) = A_3(1)$  etc. Now,  $B = A_1 A_2(2:n) A_3(2:n) \dots A_m(2:n)$  is in  $Q'(A,m)$  since if C is any permutation of any subalphabet of A of size m,  $C(j)$  is either in the j-th component of B or is the last symbol of the (j-1)th component (for  $j > 1$ ).

$$2.9 \quad M(n) \leq (n \cdot n - n + 1)$$

A simple corollary of 2.8.

$$2.10 \quad M'(n,3) = (3n-2) \quad (n \geq 3).$$

From 2.6 we get  $M'(n,3) \geq (3n-3)$ .  
From 2.8 we get  $M'(n,3) \leq (3n-2)$ .  
Suppose the lower value is obtained for an alphabet A ( $|A|=n$ ) and S is a sequence of length  $3n-3$  which is in  $Q'(n,3)$ . Now no symbol can appear only once in S for then we would have  $|S| \geq (2 \cdot M(n-1,2) + 1) = (4n-5)$  which is a contradiction for  $n \geq 3$ . Hence there must be at least 3 symbols which occur just 2 times each for a total of 6 times. However  $M(3)=7$  so there must be some permutation of these three symbols which is not a subsequence of S. This contradiction gives us the result.

2.11 Members of  $Q(1\ 2\ 3)$  of Length 7.

The following is an exhaustive list of minimum solutions for a 3 symbol alphabet. We consider, of course, only equivalence classes (with respect to the operator  $\equiv$ ).

1 2 3 1 2 1 3	1 2 3 1 2 3 1	1 2 3 1 3 2 1
1 2 3 2 1 2 3	1 2 3 2 1 3 2	
1 2 1 3 1 2 1	1 2 1 3 2 1 2	

$$2.12 \quad \forall S \in Q(A). \exists a \in A. \#(S,a) \geq |A|.$$

Use induction on the alphabet size. The case  $|A|=1$  is trivial so suppose the result holds for all alphabets of size less than n,  $|A|=n$  and  $S \in Q(A)$ . Segment S as  $TxU$  where sequences T,U and symbol x are chosen so that x does not appear in T but every other symbol of A does. Use  $A \setminus x$  to denote A minus symbol x, and we get  $U \in Q(A \setminus x)$ . Now  $|A \setminus x| = n-1$  and so we can find y such that  $\#(U,y) \geq (n-1)$ . Clearly  $\#(S,y) \geq n$ .



2.13  $\forall S \in Q^*(A, m). \text{Card}(\{a \mid a \in A \wedge \#(S, a) \geq m\}) \geq (n-m+1)$

-----  
 Let  $A$  be any alphabet,  $m$  be any integer such that  $|A| \geq m$  and  $S$  be some member of  $Q^*(A, m)$ . Select sequence  $B$  - a permutation of  $A$  such that the symbols of  $B$  are in order of decreasing frequency in  $S$ . Now take sequence  $S'$  to be the sequence formed by deleting those symbols from  $S$  which are in  $B(1:n-m)$ .  $S'$  is a member of  $Q(B(n-m+1:n))$  and so some symbol must appear at least  $m$  times in  $S'$  and hence in  $S$ .  
 Therefore,  $\#(S, B(1)) \geq \#(S, B(2)) \geq \dots \geq \#(S, B(n-m+1)) \geq m$  which gives the quoted result.

2.14  $M^*(n, m) \geq m(n-m) + M(m)$

-----  
 A corollary of 2.13 .

2.15  $M(4) = 12.$

-----  
 Take  $A$  to be the alphabet (sequence) 1 2 3 4 .

1 2 3 4 1 2 3 1 4 2 1 3  $\in Q(A)$  and so  $M(4) \leq 12$  .

Suppose  $S \in Q(A)$  and  $|S| < 12$ .  
 Compute the least integer  $j$  such that  $S(1:j)$  contains each symbol of  $A$ . Note  $j \geq 4$  and  $S(j)$  is not in  $S(1:j-1)$ .  
 Considering permutations of  $A$  which start with  $S(j)$ , we get that  $|S| \geq 3 + \#(S, S(j)) + M(3) = 10 + \#(S, S(j))$  .  
 Using  $|S| < 12$  we get  $j=4$  and  $\#(S, S(j))=1$ .  
 Therefore,  $S(4)$  appears only at position 4 of  $S$ . Now consider the permutations of  $A$  that end with  $S(4)$  and get that  $4 \geq M(3)$  which is a contradiction.

From this contradiction we see that  $M(4) \geq 12$ .

2.16  $\forall A. \forall x \in A. \exists S \in Q(A). \#(S, x) = 1$

-----  
 Suppose we are given an alphabet  $A$  and  $x$  is some symbol of  $A$ . We take the subalphabet  $A \setminus x$  and find some member  $T$  from  $Q(A \setminus x)$ . Clearly  $TxT \in Q(A)$  and also  $\#(TxT, x) = 1$ .  
 This is quite a useful result to keep in mind when pondering what properties members of  $Q(A)$  might have.

Take A to be the alphabet (sequence) 1 2 3 4 5 .

i) 1 2 3 4 5 1 2 3 4 1 5 2 3 1 4 5 2 1 3  $\in Q(A)$   
so we have  $M(5) \leq 19$  .

ii) Suppose  $S \in Q(A)$  and  $|S| < 19$  .  
Break up S as  $T y U$  (where T and U are segments of S and y is a single symbol) such that  $T y$  is the shortest initial segment of S which is in  $Q'(A, 2)$  so  $|T y| \geq M'(5, 2) = 9$ .  
Choose x in T such that xy is not a subsequence of T (this is possible otherwise S was not segmented as prescribed).

Considering members of  $P(A)$  starting with xy, get  
 $|S| \geq 9 + M(3) + \#(U, x) + \#(U, y) = 16 + \#(U, x) + \#(U, y)$ .

Now, supposing x does not appear in U, consider subsequences of S that end with x and derive the contradiction  
 $|S| \geq M(4) + 2 + M(3) = 21$ .

Conclude  $\#(U, x) \geq 1$  (and similarly  $\#(U, y) \geq 1$ ).

Reconciling inequalities, we get  $\#(U, x) = 1$ ,  $\#(U, y) = 1$ ,  $|T| = 8$ ,  
 $|U| = 9$  and  $|S| = 18$ .

In U, x and y appear just once each and so one sequence of xy and yx, call it Z, is not a subsequence of U.

Consider, then, permutations of A of the form  $\alpha Z$  and get  
 $|T| \geq M(3) + \#(T, x) + \#(T, y) \geq 9$  -- a contradiction!

We therefore conclude that  $M(5) \geq 19$ .

iii) From i) and ii) deduce  $M(5) = 19$ .

4       $M(6)=28$     and     $M(7)=39$ .  
 -      -----    ---    -----

i)      Take A to be the alphabet (sequence)    1 2 3 4 5 6 .

1 2 3 4 5 6 1 2 3 4 5 1 6 2 3 4 1 5 6 2 3 1 4 5 6 2 1 3  
 is in  $Q(A)$  so we have     $M(6) \leq 28$  .

The proof of     $M(6) \geq 28$     is given as Appendix 1 because it is  
 long and uninformative.

These two facts give the result     $M(6)=28$ .

ii)      Take A to be the alphabet    1 2 3 4 5 6 7.

1 2 3 4 5 6 7 1 2 3 4 5 6 1 7 2 3 4 5  
    1 6 7 2 3 4 1 5 6 7 2 3 1 4 5 6 7 2 1 3  
 is in  $Q(A)$  so we have     $M(7) \leq 39$  .

$M(7) \geq 39$  ( proved as appendix 2 ) and so we have     $M(7)=39$ .

# 5 Minimum Length Solutions for Alphabets of Size 4. -----

Let  $A$  be the alphabet  $a b c d$ .  
 We wish to enumerate the equivalence classes in  $Q(A)$   
 of the minimum length (ie 12). Suppose  $S \in Q(A)$  and  $|S|=12$ .

Lemma:  $\forall p \in A. \#(S,p) \geq 2$   
 $p \in A \wedge \#(S,p) = 0$  is absurd.  
 Suppose  $p \in A \wedge \#(S,p) = 1$  We have that  $S$  has the form  $UpV$ .  
 CPAF  $\alpha p$  to get  $|U| \geq M(3) = 7$ ; CPAF  $p\alpha$  to get  $|V| \geq M(3) = 7$ .  
 We immediately have the contradiction  $|S| = |UpV| \geq 15$ .

Lemma:  $\exists p. \#(S,p) = 2$   
 Suppose not. In view of above lemma,  $\forall p \in A. \#(S,p) \geq 3$  which  
 is a violation of the result 2.12 (page 3).

Supposing  $\#(S,p) = 2$ , choose  $T, U, V$  such that  $S = TpUpV$ .  
 CPAF  $p\alpha$  to get  $|UV| \geq 7$ ; CPAF  $\alpha p$  to get  $|TU| \geq 7$ .  
 Now  $|U| = |U| + (|S| - 12) = (|U| + |T| + |U| + |V| + 2) - 12 \geq 4$ .  
 Also  $|T| = |S| - 2 - |U| - |V| \leq 3$  and similarly  $|V| \leq 3$ .

Suppose  $|T| < 3$ . Thus  $\exists x \in A. \neg(x \in T) \wedge \neg(x = p)$ .  
 CPAF  $x p \alpha$  to give  $|V| \geq M(2) + \#(V,x) = 3 + \#(V,x)$ . So  $\#(V,x) = 0$ .  
 CPAF  $\alpha p x$  to give the contradiction  $|T| \geq M(2) = 3$ .  
 Hence  $|T| = 3$  and similarly  $|V| = 3$  giving  $|U| = 4$ .

Suppose  $q \in A$  and  $\neg(q = p) \wedge \#(T,q) = 0$ .  
 CPAF  $q p \alpha$  to get  $\#(V,q) = 0$ . Hence by a lemma above,  $\#(U,q) \geq 2$ .  
 CPAF  $q \alpha p$  to get the contradiction  $|U| \geq M(2) + \#(U,q) \geq 5$ .  
 Hence  $\forall q. q \in A \supset (q = p \vee \#(T,q) = \#(V,q) = 1)$ .

From this discussion we get that there are representatives of  
 all the equivalence classes of the form  
 $a b c d U d V$  where  $|U|=4, |V|=3, a \in V, b \in V, c \in V$ .

CPAF  $\alpha d$  we get  $abcU$  is in  $Q(a b c)$  and is of min. length.  
 Using result (2.11) we get 5 possibilities for  $U$ ; namely:  
 (1)  $abac$  (2)  $abca$  (3)  $acba$  (4)  $babc$  (5)  $bacb$ .

Similarly  $UV$  is in  $Q(a b c)$  and is of minimum length.  
 Performing a small amount of hand checking and using 2.11  
 again we get that there are exactly 9 equivalence classes:-

$abcd$	$abca$	$dbac$	$abcd$	$acba$	$dbca$	$abcd$	$bacb$	$dabc$
$abcd$	$abca$	$dbca$	$abcd$	$acba$	$dcab$	$abcd$	$bacb$	$dacb$
$abcd$	$abca$	$dcba$	$abcd$	$acha$	$dcba$	$abcd$	$bacb$	$dcab$

6. An  $n^2 - 2n + 4$  Construction for Alphabet of size  $n$ .  
 == =====

Given an alphabet sequence,  $A$ , of length at least three, it is asserted that the following recipe gives a sequence in  $Q(A)$ .

```
Set the sequence variable  B ← A(2:n);

Write(A);
DO (n-2) TIMES { Write(A(1)); Write( B(1:n-2) );
                  B ← (B(n-1) B(1:n-2)); };
Write(A(1)); Write(B(1));
```

The total number of symbols written =  $n + (n-2) \times (1+n-2) + 2$   
 =  $n^2 - 2n + 4$ .

We now verify that the sequence produced is indeed in  $Q(A)$ .

First note that the operation " $B \leftarrow B(n-1)B(1:n-2)$ " simply rotates the sequence of  $n-1$  symbols in  $B$ .

Next note that the first symbol of  $A$  (we will call it  $a$ ) is written exactly  $n$  times. Letting  $C$  be the result of the above construction, we segment  $C$  as follows:

$C = aJaKaLa...aYaZab$  where the  $(n-1)$  sequences  $J, K, L, \dots, Y, Z$  do not contain the symbol  $a$ .  
 For convenience we will use call  $J, K, L, \dots, Y, Z$  units and will refer to them as  $U[1], U[2], \dots, U[n-1]$ .

Now  $J$  contains all symbols  $A(2:n)$  but  $K, L, \dots, Y, Z$  each contain just  $n-2$  of the symbols of  $A(2:n)$ . However the symbol of  $A(2:n)$  that does not appear in some unit  $U[k]$  is both the last symbol of  $U[k-1]$  and follows the  $a$  that follows  $U[k]$  in  $C$ .

Let  $P$  be a permutation of  $A$ . We will show that  $P$  must be a subsequence of  $C$ .

Suppose  $a$  appears in the  $j$ th position of  $P$ . We first show that the string  $P(1:j)$  (simply  $a$  if  $j=1$ ) can be matched to the the head of  $C$   $aJaKaL...U[j-1]a$ . Trivially true if  $j=1$ . If  $j>1$  then  $P(1)$  is in  $J$ , clearly. Also if  $j>k>1$  then  $P(k)$  can be matched to  $U[k]$  if it is in that unit or else the last symbol of  $U[k-1]$ .

Similarly the  $n-j$  symbols of  $P(j+1:n)$  can be matched to  $U[j]aU[j+1]a...aU[n-1]ab$ . If  $j < k \leq n$  then  $P(k)$  will either match something in  $U[k-1]$  or the symbol which follows the  $a$  which follows  $U[k-1]$ .

7. A More General  $n^2-2n+4$  Construction.  
 --      - - - - -

It is asserted that the following algorithm, regardless of which internal choices are made, also produces a member of  $Q(A)$  of length  $n^2-2n+4$ . The proof of membership in  $Q(A)$  follows by the same method used in proving the validity of the simpler 'program'. It is also readily seen that the previous construction is a special case of this more general one.

SUBROUTINE SR1:

Write the symbol [x];  
 Write the symbol [y];

SUBROUTINE SR2:

SR1;  
 Write in any order the [n-3] symbols of A which do not include [x] or [y] or [z].  
 DO y←z AND set z to the last symbol written.

SUBROUTINE SR3:

DO SR2 [n-2] TIMES;  
 SR1;

SUBROUTINE SR4:

DO SR2 [n-3] TIMES;  
 SR1;  
 Write in any order the [n-2] symbols of A which are not [x], [y];  
 Write the symbol [x];

MAIN ROUTINE:

Write down the alphabet (A);  
 DO EITHER { x←A(1); y ← any symbol of A(2:n-1); z←A(n); }  
 OR { x ← A(2); y ← A(1); z ← A(n); };  
 DO EITHER SR3 OR SR4;

SYMBOL COUNT.

If M symbols are written each time a certain routine is obeyed then we say that the SYMBOL COUNT for that routine is M.

Symbol Count for SR1 = 2 ;  
 Symbol Count for SR2 = n-1 ;  
 Symbol Count for SR3 = (n-2)\*(n-1)+2 =  $n^2 - 3n + 4$ ;  
 Symbol Count for SR4 = (n-3)\*(n-1)+(n+1) =  $n^2 - 3n + 4$ .  
 Hence Symbol Count for total algorithm =  $n^2 - 2n + 4$  .

Note that no distinct sequences produced by this algorithm are equivalent since all such begin with a copy of the alphabet.



Note also that every sequence so produced ends with some permutation of the alphabet.

Given an alphabet  $A$ , the reversal of any sequence which is a member of  $Q(A)$  is also a member of  $Q(A)$ . It should be noted that the reverse of any sequence generated according to this construction is equivalent to some other sequence given by the construction.

8. Constructing Elements of  $Q'(A,m)$ .  
 -- -----

Section 6 contained a simple construction for generating elements of  $Q(A)$  (for given alphabet  $A$  of size  $n > 2$ ) which were of length  $n^2 - 2n + 4$ . This algorithm is now modified to generate members of  $Q'(A,m)$  (where  $2 < m \leq n$ ) of length  $mn - 2m + 4$ .

```
Set the sequence variable  B ← A(n-m+2:n);
Write(A);
DO m-2 TIMES Write( A(1:n-m+1) );
                Write( B(1:m-2) );
                B ← B(m-1)B(1:m-2);
Write( A(1:n-m+1) );
Write( B(1) );
```

The total number of symbols written is easily seen to be  
 $n + (m-2)(n-m+1 + m-2) + (n-m+1) + 1 = mn - 2m + 4$ .

Just as this algorithm is a modification of the one in section 6, the proof of the correctness of the construction is an extension of the previous proof.

This construction gives an upper bound on  $M'(n,m)$  for  $n \geq m > 2$  of  $mn - 2m + 4$  and so using this knowledge, the proposition 2.14 and the various values of  $M(4)$ ,  $M(5)$ ,  $M(6)$  &  $M(7)$  we already know, we compute the new results:-

$$\begin{aligned} M'(n,4) &= 4n-4 \\ M'(n,5) &= 5n-6 \\ M'(n,6) &= 6n-8 \\ M'(n,7) &= 7n-10 \end{aligned}$$

# 9. Discussion.

--

The construction of section 7 gives many sequences of the desired length. It gives all nine equivalence classes of sequences in  $Q(a\ b\ c\ d)$  of length 12, 128 classes in  $Q(a\ b\ c\ d\ e)$  which may or may not be all of them, and 32,400 classes from  $Q(a\ b\ c\ d\ e\ f)$ . It does NOT get all the sequences of  $Q(a\ b\ c\ d\ e\ f)$  since all the ones produced start with one copy of the alphabet however the following sequences from  $Q(a\ b\ c\ d\ e\ f)$ :

abcdebfdcabedcfbadecebdfacebd

abcdeafdcbaedcfabdecabdfbcead

(among others known) DO NOT! In fact, the second of these examples does not even end with a permutation of the alphabet.

An easy to derive lower bound on the number of classes is  $((n-3)!)^{\uparrow(n-1)}$ .

We now tabulate the known values of the functions  $M$  &  $M'$ .

m	M(m)	$m^2-2m+4$	$M'(n,m)$
1	1	3	n
2	3	4	$2n-1$
3	7	7	$3n-2$
4	12	12	$4n-4$
5	19	19	$5n-6$
6	28	28	$6n-8$
7	39	39	$7n-10$

The fact that the actual values of  $M(n)$  exactly match the  $n^2-2n+4$  figure for  $2 < n \leq 7$  make the construction relatively important. It also suggests the obvious conjecture that  $M(n)$  is exactly  $n^2-2n+4$  for all  $n > 2$ . However, there is another competing conjecture which gives exact fit at  $n=1,2$  as well as the other known values of  $M(n)$  but is more complicated:-

$$M(n) = \begin{cases} n^2 & \text{for } n=1 \\ n^2-n+1 & \text{for } 2 \leq n \leq 3 \\ n^2-2n+4 & \text{for } 4 \leq n \leq 7 \\ n^2-3n+11 & \text{for } 8 \leq n \leq 15 \\ \dots\dots\dots \\ n^2-m \times n + F(m) & \text{for } 2^m \leq n \leq 2.2^m-1 \end{cases}$$

where  $F(3)=8 \wedge F(n)=n+2 \times F(n-1)$ .

Of course, knowing whether the value for  $M(8)$  is 51 or 52 would help by eliminating one of these postulates.

It is surprising that the best lower bound we have on  $M(n)$  is  $n^2/2$  since it would appear that it is of order  $n^2$ . This conjecture is readily stated formally as:-

$$\forall k. k < 1 \Rightarrow \exists N. n > N \Rightarrow (M(n) > k \cdot n^2)$$

It should be noted that just the mechanical checking of the membership of a sequence (over alphabet A) in  $Q(A)$  is quite time-consuming. A program is available in ALGOL but (although it includes some means for pruning the tree of permutations) takes a long time to check that all permutations of the alphabet are subsequences of the given sequence. The actual times on a PDP11 are 3, 17 and 60 seconds for alphabets of sizes 8, 9 & 10 respectively.

#### REFERENCE:

1. Chvatal, V., Klarner, D.A., Knuth, D.E., "Selected Combinatorial Research Problems", Report CS 292, Computer Science Department, Stanford University, June 1972.

# APPENDIX 1. Proof of $M(6) \geq 28$ .

Take  $A$  to be an alphabet of size 6 ( $|A|=6$ ).

Moreover, suppose  $S \in Q(A)$  and  $|S| < 28$ .

Now choose sequences  $T, V$  and symbols  $x, y$  such that

a)  $Tx$  is the shortest head of  $S$  that is in  $Q'(A, 2)$ ;

b)  $yV$  is the shortest tail of  $S$  that is in  $Q'(A, 1)$ ;

Choose  $w \in T$  such that  $w \neq x \wedge \neg(w \neq T)$ .

We have immediately that  $|T| \geq 10$ ,  $|V| \geq 5$  and from consideration of the elements of  $P(A)$  of the forms  $wxx$  &  $xyy$  get  $|S| \geq |T| + 1 + M(4)$ ,  $|S| \geq |V| + 1 + M(5)$ ,  $|T| \leq 14$ ,  $|V| \leq 7$ ,  $|S| \geq 25$ .

Hence we can segment  $S$  as the sequence  $TxUyV$  and note  $10 \leq |T| \leq 14$ ,  $2 \leq |U| \leq 10$ ,  $5 \leq |V| \leq 7$ ,  $25 \leq |S| \leq 27$ .

Again CPAF  $wxx$  and get  $|UyV| \geq M(4) + 2 = 14$ . Hence (using  $|S| \leq 27$ )  $|T| \leq 12$  and (using  $|V| \leq 7$ )  $|U| \geq 6$ . Also CPAF  $xyy$  again to deduce  $|TxU| \geq M(5) + 1 = 20$ . Therefore,  $|S| \geq 20 + 1 + |V| \geq 26$  and (using  $|T| \leq 12$ )  $|U| \geq 7$ . Lastly (using  $|S| \leq 27$  and  $|TxU| \geq 20$ ),  $|V| \leq 6$ .

Suppose  $\#(U, w) = 0$ . Since  $|yV| \leq 7$  but contains all of  $A$ , there must be 5 symbols of  $yV$  which appear just once. Therefore we choose  $p, q$  such that  $p, q, x, w$  are distinct,  $\neg(pq \subset yV)$  and  $p, q$  both appear twice in  $T$ . We can do this since only one symbol of  $Tx$  can appear only once. Now CPAF  $xppq$  to get  $|T| \geq M(3) + \#(T, w) + \#(T, p) + \#(T, q) \geq 12$ . So  $|T| = 12$  and  $\#(T, w) = 1$ . Segment  $S$  as  $LwMxUyV$  noting that since  $LwMx$  is in  $P(A, 2)$  and  $\#(L, w) = 0$ ,  $|M| \geq 4$ . This gives that  $|L| \leq 7$  and  $\#(MxU, w) = 0$ .  $M(5, 2) = 9$  so we pick  $p, q$  such that  $\neg(pq \subset L)$  and  $p, q, w$  distinct. Now CPAF  $ppqw$  to get  $|yV| \geq M(3) + \#(yV, w) \geq 8$ . This contradiction gives  $\#(U, w) \geq 1$ .

Again CPAF  $wxx$  and get  $|UyV| \geq M(4) + \#(UyV, w) + \#(yV, x) \geq 15$ . Use  $|S| \leq 27$  to get  $|T| \leq 11$  and use  $|V| \leq 6$  to get  $|U| \geq 8$ .

Now let  $t \in A$  be such that  $\#(U, t) = 0$ . As above we choose  $p, q$  so that  $t, p, q$  are distinct,  $\neg(pq \subset yV)$  and  $p, q$  both appear at least twice in  $T$ . CPAF  $xtpq$  to deduce the contradiction  $|Tx| \geq M(3) + \#(Tx, t) + \#(Tx, p) + \#(Tx, q) \geq 12$  !! Hence all symbols appear at least once in  $U$ .

Yet again CPAF  $wxx$  to get  $|UyV| \geq M(4) + \#(UyV) + \#(UyV) \geq 16$ . As before deduce  $|T| \leq 10$  and  $|U| \geq 9$ . Also CPAF  $xyy$  to give  $|TxU| \geq M(5) + \#(TxU, y) \geq 21$  and then  $|S| = 27$ ,  $|V| = 5$ . We also have  $|T| = 10$ ,  $|U| = 10$  and  $\forall t. t \in A \Rightarrow t \in U$ .

The proof is concluded by deriving contradictions in the various possible cases of equality among  $w, x, y$ .

## CASE 1.

$x=y$ , and so  $S = TxUxV$ .

We know  $\#(T, x) \geq 1$  and  $\#(U, x) \geq 1$  so CPAF  $xx$  and get the contradiction  $21 = |TxU| \geq M(5) + \#(TxU, x) \geq 22$ .

CASE 2.  $x \neq y$ .

CASE 2a.  $w \neq y$  (i.e.  $w, x, y$  all distinct).

CPAF  $wx\alpha y$  to get  $|U| \geq M(3) + \#(U, w) + \#(U, x) + \#(U, y) \geq 10$

Therefore  $\#(U, w) = \#(U, x) = \#(U, y) = 1$ .

Now this gives that one of  $wx$  or  $xw$ , call it  $Z$ , is such that

$\neg(Z \subset U)$ . CPAF  $\alpha Zy$  and get  $|T| \geq M(3) + \#(T, w) + \#(T, x) + \#(T, y)$

But  $\#(T, w) + \#(T, y) \geq 3$  and so  $|T| \geq 11$  -- contradiction!!

CASE 2b.  $w = y$ .

Find the first symbol of  $V$  which is not  $x$ ; call it  $z$ .

Note that since  $yV \in P(A) \wedge |yV| = |A|$ ,  $z$  appears just once in  $V$ .

CPAF  $yx\alpha z$  to deduce  $|U| \geq M(3) + \#(U, y) + \#(U, x) + \#(U, z) \geq 10$ .

Immediately we see  $\#(U, x) = \#(U, z) = 1$  and so one of  $xz, zx$  (call it  $Z$ ) is not a subsequence of  $U$ .

CPAF  $\alpha Zy$  to get  $|T| \geq M(3) + \#(T, x) + \#(T, y) + \#(T, z)$ .

Use  $\#(T, y) + \#(T, z) \geq 3$  for the contradiction  $|T| \geq 11$ .



## APPENDIX 2. Proof of $M(7) \geq 39$ .

Take  $A$  to be an alphabet of size 7 ( $|A|=7$ ).  
Moreover, suppose  $S \in Q(A)$  and  $|S| < 39$ .

Choose sequences  $T, U, W$  and symbols  $a, b, c$  such that

- $Ta$  is the shortest head of  $S$  that is in  $Q'(A, 1)$
- $cW$  is the shortest tail of  $S$  that is in  $Q'(A, 1)$
- $TaUb$  is the shortest head of  $S$  that is in  $Q'(A, 2)$

We segment  $S$  as  $TaUbVcW$  and readily prove:

$6 \leq |T| \leq 8$ ,  $5 \leq |U| \leq 9$ ,  $8 \leq |V| \leq 18$ ,  $6 \leq |W| \leq 8$ ,  $36 \leq |S| \leq 38$ ;  
as well as  $|T| + |U| \leq 15$ .

Suppose for some  $p$  in  $A$ ,  $\#(V, p) = 0$ .

If  $p$  is the symbol  $b$ ,  $M'(6, 3) + \#(TaUb, p) \geq 18 > |TaUb|$  so we  
can choose  $q, r, s$  such that  $\text{distinct}(p, q, r, s) \wedge \neg(qrsp \leq TaUb)$   
so that  $\neg(qrsp \leq TaUbV)$ . CPAF  $qrsp\alpha$  we get a contradiction  
 $|cV| \geq 4 + M(3)$ .

Otherwise  $p, b$  are distinct and  $M'(6, 3) + \#(TaU) \geq 17 \geq |TaU|$  so  
we rechoose  $q, r, s$  such that  $\text{distinct}(p, q, r, s) \wedge \neg(qrsp \leq TaU)$   
which means  $\neg(qrsp \leq TaUbV)$ . As before get a contradiction.

Lemma 1:  $\forall x \in A$ .  $\#(V, x) \geq 1$  follows from these contradictions.

Suppose  $p \in A \wedge \text{distinct}(a, p)$ . We know  $\#(T, p) \geq 1$  and  $\#(Ub, p) \geq 1$   
and  $\#(V, p) \geq 1$  and  $\#(cW, p) \geq 1$  so conclude  $\#(S, p) \geq 4$ . Also we  
have  $\#(V, a) \geq 1$  and  $\#(cW, a) \geq 1$  so that  $\#(S, a) \geq 3$ .

We sharpen our inequalities now. CPAF  $a\alpha$  to get  $|T| \leq 7$ ,  $|S| \geq 37$ ;  
CPAF  $ab\alpha$  to get  $|T| + |U| \leq 13$ ; CPAF  $\alpha b$  to get  $|W| \leq 7$ . Hence  
 $6 \leq |T| \leq 7$ ,  $5 \leq |U| \leq 7$ ,  $13 \leq |V| \leq 18$ ,  $6 \leq |W| \leq 7$ ,  $37 \leq |S| \leq 38$ .

Suppose, in fact,  $\#(S, a) = 3$ .

We re-segment  $S$  as  $TaJaKaL$  where  $\#(TJKL, a) = 0$  and  $LcW$ .

There is at most one repeated symbol in  $T$  since  $|Ta| \leq |A| + 1$ .

Let  $z$  denote this symbol if it exists else any symbol of  $T$ .

Choose  $p, q$  such that  $\text{distinct}(p, q, a, z) \wedge \neg(pq \leq T)$ .

CPAF  $pqza\alpha$  to deduce that some subsequence  $G$  of  $KaL$  belongs  
to  $Q(A_1)$  where  $A_1$  is obtained from  $A$  by deleting  $p, q, a, z$ .  
 $|G| \geq M(3) = 7$  so some symbol of  $G$  appears at least 3 times.

So we choose  $y$  to be such a symbol and note  
 $\text{distinct}(a, y) \wedge \#(T, y) = 1 \wedge \#(KaL, y) \geq 3$ .

Now one of  $py$  and  $yp$  (call it  $Z$ ) is not a subsequence of  $T$ .

CPAF  $Zza\alpha$  to show we can choose  $x$  with the properties  
 $\text{distinct}(x, y, a) \wedge \#(T, x) = 1 \wedge \#(KaL) \geq 3$ .

Now, one of the sequences  $xy$  and  $yx$  is not a subsequence of  $T$  (call it  $Y$ ) and CPAF  $Yax$  to get

$$|KaL| \geq M(4) + \#(KaL,a) + \#(KaL,x) + \#(KaL,y) \geq 19.$$

By symmetry  $|TaJ| \geq 19$  to give the contradiction  $|S| \geq 19+19+1$ .

Lemma 2:  $\forall x \in A. \#(S,x) \geq 4$  is immediate.

Again CPAF  $ax$  to get  $|T|=6, |S|=38, \#(S,a)=4$ ;

Also CPAF  $xc$  to derive  $|W|=6, |U|+|V|=23, \#(S,c)=4$ .

Then CPAF  $abx$  to get  $|VcW| \geq M(5) + \#(VcW,a) + \#(VcW,b) \geq 23$

which leads to  $16 \leq |V| \leq 18$  and  $5 \leq |U| \leq 7$ .

Suppose that  $p,q$  are such that  $\neg(pqcV)$ . We have that

$\#(TaUb,p) + \#(TaUb,q) \geq 3$ . Now  $|TaUb| \leq 15$  and so

$$|TaUb| < M'(5,3) + \#(TaUb,p) + \#(TaUb,q). \text{ Hence we}$$

choose  $j,k,l$  such that  $\text{distinct}(j,k,l,p,q) \wedge \neg(jkl \subset TaUb)$ .

CPAF  $jklpqx$  so  $|cW| \geq M(2)+5 = 8 > |cW|$  -- a contradiction!

Thus  $\forall p \in A. \forall q \in A. \#(V,p) + \#(V,q) \geq 3$ .

In particular, letting  $z$  be the first symbol of  $cW$  which is not one of  $a,b$ ,  $\#(V,a) + \#(V,b) + \#(V,z) \geq 5$ .

CPAF  $acxz$  to get  $|V| \geq M(4) + \#(V,a) + \#(V,b) + \#(V,z) \geq 17$

Thus we have new bounds for  $U,V$ :  $5 \leq |U| \leq 6, 17 \leq |V| \leq 18$ .

We now choose sequence  $H$  and symbol  $d$  such that

$dHcW$  is the shortest tail of  $S$  in  $Q(A)$ .

By symmetry with the results for  $U$  we have that  $5 \leq |H| \leq 6$

and so we re-segment  $S$  as  $TaUbGdHcW$  where

$|T|=6, 5 \leq |U| \leq 6, 10 \leq |G| \leq 12, 5 \leq |H| \leq 6, |W|=6, |S|=38,$

$\#(S,a)=4, \#(S,c)=4$ .

Suppose  $x$  is such that  $x \neq a \wedge x \neq c \wedge \neg(e \in G)$ .

If  $x \neq b$  then CPAF  $abex$  to get

$$|dHcW| \geq M(4) + (\#(dHcW,a) + \#(dHcW,b)) + \#(dHcW,e) \geq 12+3+2$$

- a contradiction.

If  $x \neq d$  then CPAF  $xedc$  to get

$$|TaUb| \geq M(4) + (\#(TaUb,c) + \#(TaUb,d)) + \#(TaUb,e) \geq 12+3+2$$

- also a contradiction.

The remaining case is  $x=b=d$ . Lemma 1 (with  $\#(S,c)=4$ ) gives

that  $\#(TaUb,c) \leq 2$  and since there is at most one symbol in  $TaUb$

appearing 3 times, we choose  $p,q$  (not  $c$  or  $b$ ) so that  $\#(TaUb,p) \leq 2$

and  $\#(TaUb,q) \leq 2$ . Since  $M(3)=7$  there is some permutation  $Z$  of

$c,p,q$  that is not a subsequence of  $TaUb$ . CPAF  $Zbx$  to get

$$|HcW| \geq M(3) + \#(HcW,b) + \#(HcW,c) + \#(HcW,p) + \#(HcW,q) \geq 7+1+2+2+2 = 14.$$

- a contradiction.

From these 3 contradictions we get  $(x \in A \wedge x \neq a \wedge x \neq c) \supset \#(G,x) \geq 1$ .

Now suppose  $\neg(a \in G)$ . Choose  $p,q,r$  so that  $\text{distinct}(a,p,q,r)$  and

$\neg(pqr \subset dHcW)$ . CPAF  $xapqr$ . Clearly  $a \in U$  [else  $|T| \geq M(4)$ ] and so

$\#(TaUb,a) \geq 2$ . Hence

$$|TaUb| \geq M(3) + \#(TaUb,a) + \dots + \#(TaUb,r) \geq 7+2+2+2+2 = 15$$

From this contradiction we get  $\#(G,a) \geq 1$  and by symmetry  $\#(G,c) \geq 1$ .

Lemma 3:  $\forall x \in A. \#(G,x) \geq 1$  follows.

Suppose  $x \in A \wedge x \neq a \wedge x \neq c$ .  $\#(T, x) = \#(W, x) = 1$ ,  $\#(U_b, x) \geq 1$ ,  $\#(dH, x) \geq 1$   
and  $\#(S, x) \geq 1$  to yield  
Lemma 4:  $\forall x \in A. (x \neq a \wedge x \neq c) \supset \#(S, x) \geq 5$ .

Suppose distinct(a, b, c).

We first choose  $z$  to be the first symbol of  $W$  which is not  $a, b$ .

$b \neq a \wedge b \neq c$  so we have  $b \in G$ ,  $b \in dH$  giving  $\#(GdH, b) \geq 2$ .

$z \neq a \wedge z \neq c$  so we have  $z \in G$ ,  $z \in dH$  giving  $\#(GdH, z) \geq 2$ .

Also  $a \neq c$  so  $a \in dH$  and we have  $a \in G$  giving  $\#(GdH, a) \geq 2$ .

CPAF  $abxz$  to derive  $|GdH| \geq M(4) + \#(GdH, a) + \#(GdH, b) + \#(GdH, z) \geq 18$ .

We get from this that  $|U| = 5$  and also  $\#(GdH, b) = 2 = \#(GdH, z)$ .

This then gives that  $\#(S, z) = 5$  and  $\#(S, b) = 5$ .

Let  $p, q, r$  be the 3 symbols of the  $A$  which are not  $a, b, c, z$ .

$\#(S, a) + \#(S, b) + \#(S, c) + \#(S, z) = 4 + 4 + 5 + 5 = 18$

so  $\#(S, p) + \#(S, q) + \#(S, r) = 20$ .

Since no symbol appears twice in  $TaUb$ , can choose a permutation

$Z$  of  $pqr$  so that  $\neg(Z \subset TaUb)$ .

CPAF  $Z\alpha$  to get  $25 = |GdHcW| \geq M(4) + (20 - 6) = 26$  - a contradiction.

Similarly 'distinct(a, d, c)' gives a contradiction.

Lemma 5:  $\neg \text{distinct}(a, b, c) \wedge \neg \text{distinct}(a, d, c)$ .

In view of lemma 5, two important cases are  $a=c$  and  $\neg(a=c)$ .

CASE 1.  $a=c$ .

Suppose first that  $a \in U$ . Clearly  $|U| = 6$  and  $|TaUb| = 14$ .

Letting  $z$  be the first symbol of  $W$  not  $a, b$  CPAF  $abxz$  to

get  $|GdH| \geq 12 + \#(GdH, a) + \#(GdH, b) + \#(GdH, z) \geq 17$ .

But  $|GdH| = 17$  so we see  $\#(GdH, b) = 2 = \#(GdH, z)$ .

Thus  $\#(S, a) + \#(S, b) + \#(S, z) = 14$ .

Now choose  $p, q, r, s$  such that  $pqrabz$  is a permutation of  $A$  and  
 $\#(S, p) \geq \#(S, q) \geq \#(S, r) \geq \#(S, s)$ . Now since some symbol appears  
at least 7 times in  $S$ ,  $\#(S, p) \geq 7$  and  $\#(S, q) + \#(S, r) + \#(S, s) \leq 17$ .

Hence  $\#(S, s) \leq 5$  and so  $\#(S, p) + \#(S, q) + \#(S, r) \geq 19$ .

Now each of  $p, q, r$  appears exactly twice in  $TaUb$  and so

i)  $\#(GdHaW, p) + \#(GdHaW, q) + \#(GdHaW, r) \geq 13$

ii) since  $M(3) = 7$  there is a permutation of  $pqr$   
(call it  $Z$ ) such that  $\neg(Z \subset TaUb)$ .

CPAF  $Z\alpha$  to get  $24 = |GdHaW| \geq M(4) + 13 = 25$ .

This contradiction gives us  $\#(U, a) = 0$ .

Again letting  $z$  be the first symbol of  $W$  not  $a, b$  we have

$\#(GdH, a) \geq 2$ ,  $\#(GdH, b) \geq 2$ ,  $\#(GdH, z) \geq 2$  so CPAF  $abxz$  to

deduce  $|GdH| \geq 18$  and hence  $|U| = 5$  and  $\#(S, b) = \#(S, z) = 5$

Similarly,  $\#(S, d) = 5$  and  $|H| = 5$ .

$|G| = 12$  and  $\#(G, a) = \#(G, b) = 2$  so the other 5 symbols appear

a total of 8 times in  $G$ . Hence choose  $p, q$  so that  $\neg(pq \subset G)$

and distinct(a, b, p, q).  $\neg(abpq \subset TaUbG)$  so CPAF  $abpq\alpha$

to derive a contradiction  $|dHaW| \geq 7 + 3 \times 2 + 1 = 14$ .

CASE 2.  $\neg(a=c)$ .

We have  $a=b$  and  $c=d$  so Lemma 5 gives both  $b=c$  and  $d=c$ .  
Hence  $S$  looks like  $TaUbGaHbW$  with  $|T|=6$ ,  $5 \leq |U| \leq 6$ ,  $10 \leq |G| \leq 12$ ,  
 $5 \leq |H| \leq 6$ ,  $|W|=6$ ,  $\#(G,a)=\#(G,b)=1$ ,  $\#(T,b)=\#(W,a)=1$ .  
Clearly  $\#(TUH,a) = 0 = \#(UHW,b)$ .

We can write the alphabet in order of decreasing frequency in  
 $S$  as  $pqrstab$  where all except  $a,b$  occur at least 5 times and  
 $\#(S,p) \geq 7$ . Hence, as  $p,q,r,s,t$  appear a total of 30 times  
 $\#(S,t)=5$  and  $\#(S,s) \leq 6$  and  $\#(S,p)+\#(S,q)+\#(S,r) \geq 19$ .

CASE 2a:  $|U|=5$ .

Some permutation,  $Z$ , of  $pqr$  will not be a subsequence of  $TaUb$   
so CPAF  $Z\alpha$  to get  $|GaHbW| \geq 12+19-6 = 25$ .  
This gives us that  $\#(S,p)+\#(S,q)+\#(S,r) = 19$  and  $\#(S,s)=6$ .  
We then deduce  $\#(S,p)=7$ ,  $\#(S,q) = \#(S,r) = 6$ .

Now if  $z$  denotes the last symbol of  $T$  then CPAF  $z\alpha$  to get  
 $32 = |aUbGaHbW| \geq M(G) + \#(S,z) - 1$  or  $\#(S,z) \leq 5$   
But  $z=a$  so  $\#(S,z) \geq 5$  so we deduce  $z=t$ .  
Similarly the first symbol of  $W$  is  $t$ .

Recall that  $\neg(t \in TaUb)$ ,  $\#(G,a)=\#(G,b)=1$  and note  $\#(G,t)=1$ .  
CPAF  $Zab\alpha$  to deduce that  $ab \in G$ .  
CPAF  $Ztb\alpha$  to deduce that  $tb \in G$ .  
Similarly deduce that  $at \in G$ .  
i.e.  $a$  precedes  $t$  precedes  $b$  (in  $G$ ).

Suppose  $t$  is not the last symbol of  $U$ . We find  $y,z$  such that  
 $\neg(yzt \in TaUb)$  and so  $\neg(yzt \in TaUbGaH)$ . CPAF  $yzt\alpha$  for  
the contradiction by which we can conclude  $U(S)=t$ .

We have that  $S$  has the form  $T'taU'ftbGaHbW'$  where  $T't=T$ ,  
 $U'ft=U$  and  $tW'=W$  (this defines  $T'$ ,  $U'$ ,  $f$ ,  $W'$ ).  
Clearly  $f=a$ ,  $f=b$ ,  $f=t$  and so  $\#(S,f) \geq 6$ .

Now  $\neg(tf \in TaUb)$  so CPAF  $tf\alpha$  to get  $|G| \geq 7+3+\#(G,f)$ .

Suppose  $\#(G,f)=1$ . From  $\#(S,f) \geq 6$  deduce  $\#(H,f)=2$ .

Now one of  $tf, ft$  is not in  $G$  - call it  $Z$ .

CPAF  $abZ\alpha$  to get  $|aHbW| \geq 7+1+2+2+3=15$  - a contradiction.

Hence we have  $\#(G,f)=2$  and  $|G|=12$  so  $|H|=5$ .

Now let the last symbol of  $T'$  be  $g$  and suppose  $b \neq g$ .

$\neg(gb \in TaU)$  and  $\neg(ta \in G)$  so  $\neg(gbta \in TaUbG)$ .

CPAF  $gbt\alpha$  to get a contradiction.

Hence the last symbol of  $T'$  is  $b$ .

Now  $\neg(bf \in T'taU')$  but we have  $\neg(ta \in bG)$  so  $\neg(bfta \in TaUbG)$ .

CPAF  $bft\alpha$  to get  $12 = |HbW| \geq 7+1+1+2+2 = 13$ .

This last contradiction dispenses with CASE 2a.

CASE 2b:  $|H|=5$ .

The elimination of this case is similar to CASE 2a.

CASE 2c:  $|U|=5 \wedge |H|=5$ .

We have so far that  $S = TaUbGaHbW$  with  $|T|=|U|=|H|=|W|=6$   
 $|G|=10$ ,  $\#(G,a)=\#(G,b)=1$ ,  $\#(TUH,a) = \#(UHW,b) = 0$ .

Suppose first that  $\#(S,s)=5$ .

Without loss of generality suppose  $s$  precedes  $t$  in  $G$ .

$\neg(abts \leq TaUbGa)$ . Moreover if any  $p, q$  or  $r$  precedes  $s$  in  $H$   
then CPAF  $abts\alpha$  to get  $|HbW| \geq 7+1+1+4=13$  - a contradiction.

Hence only  $t$  may precede  $s$  in  $H$ .

Similarly only  $s$  may follow  $t$  in  $U$ .

Now CPAF  $atxs b$  to get  $|G| \geq 11 + \#(G,a) + \#(G,b) + \#(G,s) + \#(G,t) = 11$ .

The contradiction serves to give us  $\#(S,s)=5$ .

Hence  $\#(S,s)=6$  and  $\#(S,p)=7$ ,  $\#(S,q)=\#(S,r)=6$ .

Letting  $x$  be the duplicated symbol in  $U$  and  $y$  the duplicated  
symbol in  $H$ ,  $\#(U,x)=2$ ,  $\#(H,y)=2$ .

If  $x=y$  then  $\#(S,x) \geq 7$  so  $x=p$  and thus  $\#(G,x)=1$ .

One of  $yt, ty$  (call it  $Z$ ) is not a subsequence of  $G$ .

CPAF  $abZ\alpha$  to get  $|HbW| \geq 7+1+1+2+3=14$  - contradiction.

Else if  $y \neq p$  then  $\#(S,y)=6$  (note  $y \neq a, y \neq b, y \neq t$ ) and  $\#(G,y)=1$

One of  $yt, ty$  (call it  $Z$ ) is not a subsequence of  $G$ .

CPAF  $abZ\alpha$  to get  $|HbW| \geq 7+1+1+2+3=14$  - contradiction.

Else  $x \neq y \wedge y \neq p$  so  $x \neq p$  and  $\#(S,x)=6$ .

One of  $xt, tx$  (call it  $Z$ ) is not a subsequence of  $G$ .

CPAF  $\alpha Z ab$  to get  $|TaU| \geq 7+1+1+2+3=14$  - contradiction.

This trio of contradictions completely eliminates CASE 2c.

CASES 2a, 2b, 2c all provided contradictions as did CASE 1  
so the assumption that  $|S| < 39$  is proved impossible.

Q.E.D.